

Eichler orders, trees and Representation Fields

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Abstract

If $\mathfrak{H} \subseteq \mathfrak{D}$ are two orders in a central simple algebra \mathfrak{A} with \mathfrak{D} of maximal rank, the representation field $F(\mathfrak{D}|\mathfrak{H})$ is a subfield of the spinor class field of the genus of \mathfrak{D} which determines the set of spinor genera of orders in that genus representing the order \mathfrak{H} . Previous work have focused on two cases, maximal orders \mathfrak{D} and commutative orders \mathfrak{H} . In this work, we show how to compute the representation field $F(\mathfrak{D}|\mathfrak{H})$ when \mathfrak{D} is the intersection of a finite family of maximal orders, e.g., an Eichler order, and \mathfrak{H} is arbitrary. Examples are provided.

1 Introduction

Let K be a global field, and let \mathfrak{A} be a quaternion algebra over K . Let X be an A-curve with field of functions K as defined in §2 of [3]. For example, when K is a number field we can assume $X = \text{Spec}(\mathcal{O}_{K,S})$ for a finite set $S = X^c$ containing the infinite places, but we also include the case in which X is an affine or projective curve over a finite field, where in the latter case we set $S = \emptyset$. In [3] we computed a representation field that determines the set of spinor genera of maximal orders that contain a conjugate of a given commutative order. In fact, the definition of genera, spinor genera, spinor class fields, and representation fields, given in [3] for maximal order, has a straightforward extension for orders of maximal rank, extending the usual definition when K is a number field [1]. When strong approximation holds, spinor genera coincide with conjugacy classes, just as in the number field case. The existence of a representation field F for \mathfrak{H} implies that the

proportion of conjugacy classes in the genus $\mathbb{O} = \text{Gen}(\mathfrak{D})$ representing \mathfrak{H} is $[F : K]^{-1}$. This fact was first studied by Chevalley [6] when \mathfrak{A} is a matrix algebra of arbitrary dimension, \mathfrak{D} is a maximal order, and \mathfrak{H} is the maximal order in a maximal subfield in \mathfrak{A} . In more recent times, several authors have studied the problem in the more restricted case of quaternion algebras. The following table summarizes the main results in this area:

| Year | Ref. | \mathfrak{D} | \mathfrak{H} | other |
|------|------|----------------|----------------|------------------------------------------------------------------------------|
| 1999 | [7] | maximal | commutative | |
| 2004 | [8] | Eichler | commutative | |
| 2004 | [5] | Eichler | commutative | Proves existence with no restriction on \mathfrak{D} |
| 2008 | [2] | maximal | Eichler | Finds counterexample with $\dim_K \mathfrak{A} = 9$ |
| 2008 | [10] | EOSFL | commutative | Considers optimal embeddings |
| 2011 | [9] | | commutative | Assumes some technical conditions on the pair $(\mathfrak{D}, \mathfrak{H})$ |

Here EOSFL means Eichler order of square free level. In 2011, the representation field was computed for commutative orders \mathfrak{H} in maximal orders \mathfrak{D} of central simple algebras of arbitrary dimension [3]. The commutativity condition on \mathfrak{H} is only necessary in a technical step, and in fact the method in [3] allows the computation of spinor class fields for other interesting families of orders, like cyclic orders [4]. However, the condition that \mathfrak{D} is maximal is essential in this computations and a generalization to arbitrary orders of maximal rank seems unlikely at this point.

To fix ideas, we say that two X -orders \mathfrak{D} and \mathfrak{D}' , of maximal rank in \mathfrak{A} , are in the same genus, if $\mathfrak{D}' = a_\varphi \mathfrak{D} a_\varphi^{-1}$ for some local element a_φ in every completion \mathfrak{A}_φ^* . Equivalently, \mathfrak{D} and \mathfrak{D}' are in the same genus if $\mathfrak{D}' = a \mathfrak{D} a^{-1}$ for some adelic element $a \in \mathfrak{A}_\mathbb{A}^*$. Similarly, two X -orders \mathfrak{D} and \mathfrak{D}' , of maximal rank in \mathfrak{A} , are in the same spinor genus, if $\mathfrak{D}' = b \mathfrak{D} b^{-1}$ for some local element $b = rc \in \mathfrak{A}_\mathbb{A}^*$, where $r \in \mathfrak{A}^*$ is a global element, while c is an element with trivial reduced norm. The spinor class field $\Sigma = \Sigma(\mathbb{O})$ for a genus \mathbb{O} of orders of maximal rank is defined as the class field corresponding to $K^*H(\mathfrak{D})$, where \mathfrak{D} is an order in \mathbb{O} , and $H(\mathfrak{D})$ is the group of reduced norms of adelic elements stabilizing \mathfrak{D} by conjugation. This definition is independent of the choice of $\mathfrak{D} \in \mathbb{O}$, since the reduced norm is invariant under conjugation. The importance of this field lies in the existence of a map

$$\rho : \mathbb{O} \times \mathbb{O} \rightarrow \text{Gal}(\Sigma/K),$$

with the following properties:

1. \mathfrak{D} and \mathfrak{D}' are conjugate if and only if $\rho(\mathfrak{D}, \mathfrak{D}') = \text{Id}_\Sigma$,
2. $\rho(\mathfrak{D}, \mathfrak{D}'') = \rho(\mathfrak{D}, \mathfrak{D}')\rho(\mathfrak{D}', \mathfrak{D}'') \quad \forall (\mathfrak{D}, \mathfrak{D}', \mathfrak{D}'') \in \mathbb{O}^3$,

When \mathfrak{H} is a suborder of \mathfrak{D} (or some other order in the genus of \mathfrak{D}) the representation field $F = F(\mathfrak{D}|\mathfrak{H})$ is the class field of the set $K^*H(\mathfrak{D}|\mathfrak{H})$, where

$$H(\mathfrak{D}|\mathfrak{H}) = \{N(a) | a \in \mathfrak{A}_\mathbb{A}^*, \text{ and } a\mathfrak{H}a^{-1} \subseteq \mathfrak{D}\},$$

if this set turns out to be a group. It has the property that \mathfrak{H} embeds into an order $\mathfrak{D}' \in \mathbb{O}$ if and only if $\rho(\mathfrak{D}, \mathfrak{D}')$ is trivial on F . When $F(\mathfrak{D}|\mathfrak{H})$ is defined the number of spinor genera representing \mathfrak{H} divides the total number of spinor genera. This is not always the case in algebras of higher dimension [2]. The proof of all these facts is a word-by-word transliteration of the proof for maximal orders ([3], §2).

The purpose of the current work is to give a formula for the representation field $F(\mathfrak{D}|\mathfrak{H})$ whenever $\mathfrak{D} = \mathcal{O}_X + I\mathfrak{D}_0$ for an Eichler order \mathfrak{D}_0 in a quaternion algebra \mathfrak{A} , and an integral ideal, i.e., a 1-dimensional lattice, I in K . Our result has no restriction on the sub-order \mathfrak{H} :

Theorem 1.1. *Let $\mathfrak{D} = \mathcal{O}_X + I\mathfrak{D}_0$ in the preceding notations. Then the spinor class field of \mathfrak{D} is the largest field Σ satisfying the following conditions:*

1. Σ/K is an extension of exponent 2 unramified at the finite primes.
2. Σ/K splits completely at every place \wp satisfying one of the following conditions:
 - (a) \wp is non-archimedean and $\wp \notin X$,
 - (b) \wp is archimedean and \mathfrak{A}_\wp is a matrix algebra,
 - (c) $\wp \in X$ and \mathfrak{A}_\wp is a division algebra,
 - (d) the level d_\wp of \mathfrak{D}_0 at \wp is odd.

Furthermore, for any suborder $\mathfrak{H} \subseteq \mathfrak{D}$, the representation field is the largest subfield splitting completely at every place \wp where \mathfrak{H} embeds locally in $\mathcal{O}_X + I\mathfrak{D}_1$ for an Eichler order \mathfrak{D}_1 whose level at \wp is strictly larger than d_\wp .

The description of the class group of Σ , in the specific case of Eichler orders and without the language of spinor class fields, appears already in Corollary III.5.7 in [12]. Here we obtain this computation with no effort as

a consequence of the general computation of relative spinor images in terms of branches obtained in §3. The condition in the last statement of Theorem 1.1 can easily be decided by the methods described in §4. As a consequence, we generalize the results for Eichler orders in [8] or [5] in this manner (§6).

The importance of the orders considered here lays in the fact that they are the only orders that can be written as the intersection of a family of maximal orders. In fact, as a byproduct of our work on these orders, we prove in §5 the following result:

Theorem 1.2. *For an order of maximal rank \mathfrak{D} , the following conditions are equivalent:*

1. $\mathfrak{D} = \mathcal{O}_X + I\mathfrak{D}_0$ for an Eichler order \mathfrak{D}_0 and an integral ideal I such that $I_{\wp} = \mathcal{O}_{\wp}$ whenever \mathfrak{A}_{\wp} is a division algebra.
2. \mathfrak{D} is the intersection of a finite family of maximal orders.
3. \mathfrak{D} is the intersection of three maximal orders.

Note that if \mathfrak{D} is an arbitrary order of maximal rank and \mathfrak{D}' is the intersection of the maximal orders containing \mathfrak{D} , we have $F(\mathfrak{D}'|\mathfrak{H}) \subseteq F(\mathfrak{D}|\mathfrak{H})$, so at least an effective lower bound for the representation field can be obtained by the methods of the present paper for any order of maximal rank. This can be used in some cases to prove selectivity.

2 Trees and branches

In all of this section, we let K be a local field with ring of integers \mathcal{O}_K and maximal ideal $m_K = \pi\mathcal{O}_K$. Let \mathfrak{T} be the Bruhat-Tits tree of $PGL_2(K)$, i.e., the vertices of \mathfrak{T} are the maximal orders in $\mathbb{M}_2(K)$, while two of them, \mathfrak{D}_1 and \mathfrak{D}_2 are joined by an edge if and only if $[\mathfrak{D}_1 : \mathfrak{D}_1 \cap \mathfrak{D}_2] = [\mathcal{O}_K : m_K]$ ([11], §II.1). Recall that every maximal order has the form $\mathfrak{D}_{\Lambda} = \{x \in \mathbb{M}_2(K) | x\Lambda \subseteq \Lambda\}$ for some lattice $\Lambda \subseteq K^2$, and $\mathfrak{D}_{\Lambda} = \mathfrak{D}_{\Lambda'}$ if and only if $\Lambda' = \lambda\Lambda$ for some $\lambda \in K^*$. For any order \mathfrak{H} (of arbitrary rank) in $\mathbb{M}_2(K)$, we define $\mathfrak{H}^{[s]} = \mathcal{O}_K + \pi^s \mathfrak{H}$.

Proposition 2.1. *For every non-negative integers s and t , the following properties hold:*

1. If $\mathfrak{H}^{[t]} \subseteq \mathfrak{H}_1^{[t]}$, for some integer $t \geq 0$, then $\mathfrak{H}^{[s]} \subseteq \mathfrak{H}_1^{[s]}$ for any integer $s \geq 0$.

$$2. (\mathfrak{H}_1 \cap \mathfrak{H}_2)^{[s]} = \mathfrak{H}_1^{[s]} \cap \mathfrak{H}_2^{[s]}.$$

$$3. (\mathfrak{H}^{[s]})^{[t]} = \mathfrak{H}^{[s+t]}.$$

Proof. Property (3) and the case $t = 0$ of the first statement are trivial, and therefore $(\mathfrak{H}_1 \cap \mathfrak{H}_2)^{[s]} \subseteq \mathfrak{H}_1^{[s]} \cap \mathfrak{H}_2^{[s]}$. Next we prove $\mathfrak{H}_1^{[s]} \cap \mathfrak{H}_2^{[s]} \subseteq (\mathfrak{H}_1 \cap \mathfrak{H}_2)^{[s]}$. Assume $\lambda + \pi^s \rho = \mu + \pi^s \sigma$, where $\lambda, \mu \in \mathcal{O}_K$, $\rho \in \mathfrak{H}_1$, and $\sigma \in \mathfrak{H}_2$. Then ρ and σ commute, whence $\sigma - \rho = \pi^{-s}(\lambda - \mu)$ is integral over \mathcal{O}_K . We conclude that $\sigma - \rho \in \mathcal{O}_K$, whence (2) follows. The general proof of (1) is similar. \square

For any order $\mathfrak{H} \subseteq \mathbb{M}_2(K)$ and any integer $r \geq 0$ we define the set

$$S_r(\mathfrak{H}) = \{\mathfrak{D} \in V(\mathfrak{T}) \mid \mathfrak{H} \subseteq \mathfrak{D}^{[r]}\},$$

and call it the r -branch of \mathfrak{H} . Next result follows easily from the definition:

Proposition 2.2. *The following properties hold for any order $\mathfrak{H} \subseteq \mathbb{M}_2(K)$:*

1. $S_0(\mathfrak{H})$ is non-empty.
2. For every triple of integers (r, k, t) we have $S_{r+t}(\mathfrak{H}^{[k+t]}) = S_r(\mathfrak{H}^{[k]})$.
3. If $\mathfrak{H} \subseteq \mathfrak{H}'$, then $S_r(\mathfrak{H}) \supseteq S_r(\mathfrak{H}')$ for any integer r .
4. If \mathfrak{H}' is the intersection of a family of maximal orders, and if $S_0(\mathfrak{H}) \supseteq S_0(\mathfrak{H}')$, then $\mathfrak{H} \subseteq \mathfrak{H}'$.
5. $\mathfrak{H} \subseteq \mathfrak{D}^{[r]}$ for some Eichler order \mathfrak{D} of level d if and only if $S_r(\mathfrak{H})$ contains two vertices at distance d .

A set of vertices in a tree is said to be connected if for every pair of its points it contains every vertex in the unique path joining them. Let $\mathfrak{D}, \mathfrak{D}' \in S_r(\mathfrak{H})$. Then $\mathfrak{H} \subseteq \mathcal{O}_K + \pi^r(\mathfrak{D} \cap \mathfrak{D}')$. Now recall that the Eichler order $\mathfrak{D} \cap \mathfrak{D}'$ is contained in every order in the path joining \mathfrak{D} and \mathfrak{D}' . Next result follows:

Proposition 2.3. *For any order \mathfrak{H} and any integer r , the branch $S_r(\mathfrak{H})$ is connected.*

Proposition 2.4. *For any order \mathfrak{H} and any integer t , the branch $S_0(\mathfrak{H}^{[t]})$ contains exactly the maximal orders at a distance $\leq t$ of some maximal order in $S_0(\mathfrak{H})$.*

Proof. It suffices to prove the case $t = 1$ and use (3) in Proposition 2.1. Now observe that two maximal orders \mathfrak{D} and \mathfrak{D}_1 are neighbors if and only if, in some basis $\{v, w\}$ they corresponds to the lattices $\Lambda = \mathcal{O}_K v + \mathcal{O}_K w$ and $\Lambda_1 = \mathcal{O}_K v + \pi \mathcal{O}_K w$ respectively. The fact that $\mathfrak{D}_1 \in S_0(\mathfrak{H})$ means that for every element $h \in \mathfrak{H}$ we have $h\Lambda_1 \subseteq \Lambda_1$. Note that $\pi h\Lambda_1 \subseteq \Lambda_1$, and $h(\pi\Lambda) \subseteq h\Lambda_1 \subseteq \Lambda_1 \subseteq \Lambda$ for every $h \in \mathfrak{H}$. Then $\mathfrak{H}^{[1]}\Lambda_1 \subseteq \Lambda_1$ and $\mathfrak{H}^{[1]}\Lambda \subseteq \Lambda$, whence sufficiency follows. On the other hand, assume that $\mathfrak{D} \in S_0(\mathfrak{H}^{[1]})$, and set $\mathfrak{D} = \mathfrak{D}_\Lambda$. Then for every $h \in \mathfrak{H}$, we have $\pi h \in \mathfrak{H}^{[1]}$, whence $\pi h\Lambda = \Lambda$, and therefore the \mathfrak{H} -invariant lattice $\Lambda' = \mathfrak{H}\Lambda$ is contained in $\pi^{-1}\Lambda$. There are three possibilities:

1. If $\Lambda' = \Lambda$, then $\mathfrak{D} \in S_0(\mathfrak{H})$.
2. If $[\mathfrak{H}' : \mathfrak{H}] = |\mathcal{O}_K/m_K|$, then the maximal order \mathfrak{D}' corresponding to Λ' is a neighbor of \mathfrak{D} and $\mathfrak{D}' \in S_0(\mathfrak{H})$.
3. If $\Lambda' = \pi^{-1}\Lambda$, then $\pi^{-1}\Lambda$, and therefore also Λ , are \mathfrak{H} -invariant. This is a contradiction, so this case cannot hold.

The result follows. \square

Corollary 2.1. *Let \mathfrak{D} be a maximal order. Then $S_0(\mathfrak{D}^{[t]})$ is the set of orders at a distance at most t from \mathfrak{D} .*

In particular, a maximal order contains $\mathfrak{D}^{[t]}$ if and only if it is located at a distance not bigger than t in the graph. In fact, a stronger statement is true.

Lemma 2.1. *$\mathfrak{D}^{[t]}$ is the intersection of all orders at a distance at most t from \mathfrak{D} .*

Proof. Let \mathfrak{D}' be the intersection of all maximal orders at a distance at most t from \mathfrak{D} . From the preceding corollary, $\mathfrak{D}^{[t]} \subseteq \mathfrak{D}'$. We prove the converse. Let $u \in \mathfrak{D}'$, i.e., u is contained in every maximal order at a distance at most t from \mathfrak{D} . Let \mathfrak{D}_1 and \mathfrak{D}_2 be two orders at distance t from \mathfrak{D} , such that the shorter path between \mathfrak{D}_1 and \mathfrak{D}_2 passes through \mathfrak{D} . In some choice of coordinates, the orders \mathfrak{D}_1 , \mathfrak{D} , and \mathfrak{D}_2 are respectively:

$$\mathfrak{D}_1 = \begin{pmatrix} \mathcal{O}_K & \pi^t \mathcal{O}_K \\ \mathcal{O}_K & \mathcal{O}_K \end{pmatrix}, \quad \mathfrak{D} = \begin{pmatrix} \mathcal{O}_K & \mathcal{O}_K \\ \mathcal{O}_K & \mathcal{O}_K \end{pmatrix}, \quad \mathfrak{D}_2 = \begin{pmatrix} \mathcal{O}_K & \mathcal{O}_K \\ \pi^t \mathcal{O}_K & \mathcal{O}_K \end{pmatrix}.$$

It follows that $u \equiv \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \pmod{\pi^t}$ for some $a, b \in \mathcal{O}_K$. Since conjugation by the element $h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ leaves invariant \mathfrak{D} , and therefore also the set of maximal orders at a distance no bigger than t from \mathfrak{D} , the element $huh^{-1} \equiv \begin{pmatrix} a & b-a \\ 0 & b \end{pmatrix} \pmod{\pi^t}$ also belongs to \mathfrak{D}' . We conclude that $b \equiv a \pmod{\pi^t}$, and therefore $u \in \mathfrak{D}^{[t]}$. \square

A maximal order $\mathfrak{D} \in S_0(\mathfrak{H})$ such that every maximal order at a distance at most t from \mathfrak{D} belongs to $S_0(\mathfrak{H})$ is said to be t -deep in $S_0(\mathfrak{H})$.

Corollary 2.2. *$S_r(\mathfrak{H})$ is the set of r -deep maximal orders in $S_0(\mathfrak{H})$.*

We conclude that in order to compute $S_r(\mathfrak{H}^{[t]})$, it suffices to compute $S_0(\mathfrak{H})$.

3 relative spinor images

In all of this section K is a local field with maximal order \mathcal{O}_K , uniformizing parameter π , and absolute value $x \mapsto |x|_K$. Let \mathfrak{A} be a split quaternion K -algebra, i.e., $\mathfrak{A} \cong \mathbb{M}_2(K)$, and let $N : \mathfrak{A}^* \rightarrow K^*$ be the reduced norm. For any pair of orders $\mathfrak{H} \subseteq \mathfrak{D}$ in \mathfrak{A} define the local relative spinor image $H(\mathfrak{D}|\mathfrak{H}) \subseteq K^*$ by

$$H(\mathfrak{D}|\mathfrak{H}) = \{N(u) | u\mathfrak{H}u^{-1} \subseteq \mathfrak{D}\},$$

and let $H(\mathfrak{D}) = H(\mathfrak{D}|\mathfrak{D})$. Recall that $H(\mathfrak{D}|\mathfrak{H}) = H(\mathfrak{D})H(\mathfrak{D}|\mathfrak{H})H(\mathfrak{H})$ [1]. Note that for any order \mathfrak{D} and any invertible element u in \mathfrak{A} , we have $u\mathfrak{D}^{[r]}u^{-1} = (u\mathfrak{D}u^{-1})^{[r]}$ for any non-negative integer r , and the correspondence $\mathfrak{D} \mapsto \mathfrak{D}^{[r]}$ is injective and preserves inclusions. It follows that $H(\mathfrak{D}^{[r]}|\mathfrak{H}^{[r]}) = H(\mathfrak{D}|\mathfrak{H})$. In particular $H(\mathfrak{D}^{[r]}) = H(\mathfrak{D})$.

Let $\mathfrak{D} = \mathfrak{D}_1 \cap \mathfrak{D}_2$ be an Eichler order of level d . In other words, $[\mathfrak{D}_i : \mathfrak{D}] = |\pi|_K^d$ for $i = 1, 2$. Note that d is the distance between \mathfrak{D}_1 and \mathfrak{D}_2 in the Bruhat-Tits tree \mathfrak{T} of $\mathrm{PGL}_2(K)$. Let \mathfrak{H} be a suborder of $\mathfrak{D}^{[r]}$. Let $S_r(\mathfrak{H})$ be as defined in the preceding section. It follows that $S_r(\mathfrak{H})$ has two points, namely \mathfrak{D}_1 and \mathfrak{D}_2 , at distance d . Note that $H(\mathfrak{D}^{[r]}) = H(\mathfrak{D}) \supseteq \mathcal{O}_K^* K^{*2}$, since the Eichler order \mathfrak{D} contains a conjugate of the diagonal matrix $\mathrm{diag}(1, u)$ for every unit $u \in \mathcal{O}_K^*$. In particular, $H(\mathfrak{D}^{[r]})$ is either $\mathcal{O}_K^* K^{*2}$ or K^* .

Lemma 3.1. *$H(\mathfrak{D}^{[r]}|\mathfrak{H}) = K^*$ if and only if there exist another pair $(\mathfrak{D}_3, \mathfrak{D}_4)$ in $S_r(\mathfrak{H}) \times S_r(\mathfrak{H})$ also at distance d such that the distance between \mathfrak{D}_1 and \mathfrak{D}_3 is odd.*

Proof. Assume the condition is satisfied, and take $\sigma \in GL_2(K)$ such that $\sigma\mathfrak{D}_1\sigma^{-1} = \mathfrak{D}_3$ and $\sigma\mathfrak{D}_2\sigma^{-1} = \mathfrak{D}_4$. Then $\mathfrak{H} \subseteq (\mathfrak{D}_3^{[r]} \cap \mathfrak{D}_4^{[r]}) = \sigma\mathfrak{D}^{[r]}\sigma^{-1}$. It follows that σ is a local generator for $\mathfrak{D}^{[r]}|\mathfrak{H}$. Since the distance between \mathfrak{D}_1 and \mathfrak{D}_3 is odd and $\sigma\mathfrak{D}_1\sigma^{-1} = \mathfrak{D}_3$, the reduced norm $n(\sigma)$ has odd valuation (Corollary to Prop. 1 in §II.1.2 of [11]). Assume now that $H(\mathfrak{D}^{[r]}|\mathfrak{H}) = K^*$, so there must exist a generator σ with reduced norm of odd valuation. Then let $\mathfrak{D}_3 = \sigma\mathfrak{D}_1\sigma^{-1}$ and $\mathfrak{D}_4 = \sigma\mathfrak{D}_2\sigma^{-1}$. Since σ is a generator, we must have $\mathfrak{H} \subseteq \sigma\mathfrak{D}^{[r]}\sigma^{-1} = (\mathfrak{D}_3 \cap \mathfrak{D}_4)^{[r]}$, while the fact that the reduced norm of σ has odd valuation means that the distance from \mathfrak{D}_1 to \mathfrak{D}_3 is also odd (Corollary to Prop. 1 in §II.1.2 of [11]). \square

Corollary 3.1. *If the level of $\mathfrak{D} = \mathfrak{D}_1 \cap \mathfrak{D}_2$ is odd then $H(\mathfrak{D}^{[r]}|\mathfrak{H}) = K^*$.*

Proof. It suffices to define $(\mathfrak{D}_3, \mathfrak{D}_4) = (\mathfrak{D}_2, \mathfrak{D}_1)$. \square

Corollary 3.2. *If the level of \mathfrak{D} is smaller than the diameter of $S_r(\mathfrak{H})$, then $H(\mathfrak{D}^{[r]}|\mathfrak{H}) = K^*$.*

Proof. Let $\mathfrak{E}_0 - \mathfrak{E}_1 - \cdots - \mathfrak{E}_{d+1}$ a path of length $d+1$ in $S_r(\mathfrak{H})$. Then either $(\mathfrak{D}_3, \mathfrak{D}_4) = (\mathfrak{E}_0, \mathfrak{E}_d)$ or $(\mathfrak{D}_3, \mathfrak{D}_4) = (\mathfrak{E}_1, \mathfrak{E}_{d+1})$ satisfies the hypotheses of the previous lemma. \square

If \mathfrak{D} is maximal, so that $d = 0$, then $H(\mathfrak{D}^{[r]}|\mathfrak{H}) = K^*$ as soon as \mathfrak{H} is contained in a second order of the form $\mathfrak{D}_1^{[r]}$ for \mathfrak{D}_1 maximal. On the other hand, if $S_r(\mathfrak{H})$ contains exactly one point, then the condition of Lemma 3.1 cannot be satisfied. Next result follows:

Corollary 3.3. *If \mathfrak{D} is maximal, then $H(\mathfrak{D}^{[r]}|\mathfrak{H}) = K^*$ if and only if $\mathfrak{H} \subseteq \mathfrak{D}_0^{[r]}$ for a maximal order \mathfrak{D}_0 implies $\mathfrak{D}_0 = \mathfrak{D}$. In particular, $H(\mathfrak{D}|\mathfrak{H}) = K^*$ if and only if \mathfrak{H} is contained in a unique maximal order.*

If \mathbb{H} is the finite algebra defined in §3 of [3], then in the quaternionic case the irreducible representations of \mathbb{H} have dimensions 1 or 2, and in the latter case this representation is unique. This can happen only if \mathbb{H} contains the unique quadratic extension of the finite field \mathbb{F}_q . From the preceding corollary and Lemma 3.3 in [3], next result follows:

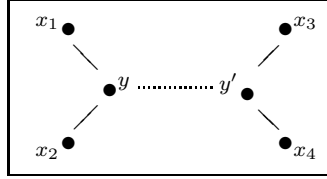
Corollary 3.4. *If \mathfrak{D} is maximal, then \mathfrak{H} is contained in a unique maximal order if and only if it contains the maximal order of an unramified extension.*

Proposition 3.1. *Let d be an even integer. If d is the diameter of $S_r(\mathfrak{H})$, and \mathfrak{D} has level d , then $H(\mathfrak{D}_r|\mathfrak{H}) = \mathcal{O}_K^* K^{*2}$.*

This result follows from next lemma:

Lemma 3.2. *Let T be a finite tree with even diameter d . Let (x_1, x_2) and (x_3, x_4) be two pairs of vertices at distance d . Then the distance from x_1 to x_3 is even.*

Proof. Denote by ρ the usual distance in the graph T . Consider the path joining (x_1, x_2) and the path joining (x_3, x_4) . If no edge of these paths is a common edge, there must be a unique (possibly empty) minimal path joining two of their vertices, say y and y' , as the figure show:



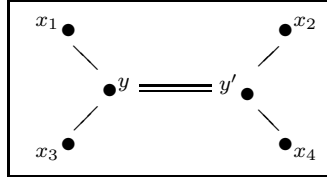
Note that

$$\rho(x_1, x_4) + \rho(x_3, x_2) + \rho(x_1, x_3) + \rho(x_2, x_4) = 4[d + \rho(y, y')],$$

but d is the diameter, whence $y = y'$ and every distance on the left is equal to d . It follows that

$$\rho(x_1, y) = \frac{1}{2} [\rho(x_1, x_4) + \rho(x_1, x_3) - \rho(x_3, x_4)] = d/2.$$

the same argument holds for all other extremes. Now assume that the path joining y and y' is common to both, the path joining x_1 and x_2 , and the path joining x_3 and x_4 like in next picture:



This includes the possibility that y or y' coincide with one of the endpoints. Then $\rho(x_3, x_2) \leq d = \rho(x_1, x_2)$, whence $\rho(x_3, y) \leq \rho(x_1, y)$, and by symmetry, $\rho(x_3, y) = \rho(x_1, y)$. The result follows. \square

By setting $\mathfrak{H} = \mathfrak{D}^{[r]}$, we obtain from Corollary 3.1 and the preceding proposition:

Corollary 3.5. *$H(\mathfrak{D}^{[r]})$ is $\mathcal{O}_K^* K^{*2}$ if d is even and K^* otherwise.*

Proof of Theorem 1.1 Recall that The spinor class field of an order \mathfrak{D} is the class field of the group $K^*H(\mathfrak{D}) \subseteq J_K$. Write $H(\mathfrak{D}) = \prod_{\wp} H_{\wp}(\mathfrak{D})$ in terms of the local spinor images defined in this section. Now the first statement is a consequence of the following observations:

1. $J_K^2 \subseteq H(\mathfrak{D})$ since scalar matrices normalize any order.
2. If $\wp \notin X$, then $H_{\wp}(\mathfrak{D})$ contains $N(\mathfrak{A}_{\wp}^*) = K_{\wp}^*$, unless \wp is a real place that is ramified for \mathfrak{A} , in which case $N(\mathfrak{A}_{\wp}^*) = K_{\wp}^+$.
3. Similarly, at places \wp where \mathfrak{A}_{\wp} is a division algebra we have $H_{\wp}(\mathfrak{D}) = H_{\wp}(\mathfrak{D}_0) = N(\mathfrak{A}_{\wp}^*) = K_{\wp}^*$, since \mathfrak{D}_0 is the unique maximal order.
4. If \wp is a place where the level of \mathfrak{D}_0 is odd, then $H_{\wp}(\mathfrak{D}) = K_{\wp}^*$ by Corollary 3.5. By the same result $H_{\wp}(\mathfrak{D}) = \mathcal{O}_{\wp}^* K_{\wp}^{*2}$ at the remaining places.

The second statement is proven analogously by using Corollary 3.2 and Proposition 3.1. \square

4 Branches of commutative orders and computations

We note that if an order \mathfrak{H} is expressed in terms of a set of generators $\{a_1, \dots, a_s\}$, then a maximal order contains \mathfrak{H} if and only if it contains all the generators. It follows that

$$S_r(\mathfrak{H}) = \bigcap_{i=1}^s S_r(\mathcal{O}_K[a_i]).$$

It suffices therefore to compute $S_r(\mathfrak{H})$ for commutative orders. Recall that we can always assume $r = 0$ in these computations.

Lemma 4.1. *If L is a semisimple commutative subalgebra of \mathfrak{A} , and Ω is an order in L , then $\Omega = \mathcal{O}_L^{[t]}$ for some non-negative integer t .*

Proof. Note that for any element $a \in \mathfrak{A}$ that is integral over \mathcal{O}_K , an element of the form $a + \lambda$ with $\lambda \in K$ is integral over \mathcal{O}_K if and only if $\lambda \in \mathcal{O}_K$. We conclude that Ω is completely determined by its image in the abelian group L/K . The result follows since every order is contained in a maximal order and the K -vector space L/K is one-dimensional. \square

Proposition 4.1. *Let L be a quadratic extension of K . Then $S_0(\mathcal{O}_L) = S_0(\mathfrak{D})$ where \mathfrak{D} is a maximal order if the extension L/K is unramified and an Eichler order of level 1 otherwise. When L is isomorphic to $K \times K$, then $S_0(\mathcal{O}_L)$ is a maximal path in the tree.*

Proof. If \mathcal{O}_L is the maximal order in a quadratic field extension L/K , then the maximal orders containing \mathcal{O}_L are in correspondence with the \mathcal{O}_L -invariant lattices in the (unique) two dimensional representation of L as a K -algebra. By identifying K^2 with L as vector spaces, we obtain that the maximal orders containing \mathcal{O}_L are in correspondence with the fractional ideals on L modulo K^* -multiplication. In other words, \mathcal{O}_L is contained in a unique maximal order if L/K is unramified and exactly two (necessarily neighbors) if L/K is ramified. The last statement is similar. If $L \cong K \times K$, the fractional ideals are of the form $\pi^s \mathcal{O}_K \times \pi^t \mathcal{O}_K$. It is readily seen that the corresponding orders lie in a maximal path. \square

Corollary 4.1. *If Ω is a commutative order contained in a field, then any branch $S_r(\Omega)$ is the set of orders at distance not exceeding t from either a vertex or two neighboring vertices, for some non-negative integer t . If Ω is a commutative order contained in an algebra isomorphic to $K \times K$, then any branch $S_r(\Omega)$ is the set of orders at distance not exceeding t from some maximal path in the tree, for some non-negative integer t .*

Proposition 4.2. *Let $\Omega = \mathcal{O}_K[a]$ where $a \neq 0$ and $a^2 = 0$. Let Λ be a lattice of the form $\Lambda = \mathcal{O}_K v + \mathcal{O}_K w$, where $av = w$ and $aw = 0$. Then if \mathfrak{D}_i is the maximal order corresponding to the lattice $\pi^i \mathcal{O}_K v + \mathcal{O}_K w$, then $S_0(\Omega)$ is the set of maximal orders at a distance at most i from \mathfrak{D}_i for some $i \geq 0$.*

Proof. Without loss of generality we can assume $(v, w) = (e_1, e_2)$ is the canonical basis. In particular

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } \mathfrak{D}_i = \begin{pmatrix} \mathcal{O}_K & \pi^i \mathcal{O}_K \\ \pi^{-i} \mathcal{O}_K & \mathcal{O}_K \end{pmatrix}.$$

It follows that a is contained in each of the orders \mathfrak{D}_i for $i \geq 0$. Now, let $h = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$. Then $h^{-1} \mathfrak{D}_i h = \mathfrak{D}_{i-1}$, while $h^{-1} \Omega h = \Omega^{[1]}$. It follows that $h^{-1} S_0(\Omega) h = S_0(\Omega^{[1]})$, and by iteration $h^{-t} S_0(\Omega) h^t = S_0(\Omega^{[t]})$ for every $t \geq 1$. Note now that \mathfrak{D}_1 is the unique neighbor of \mathfrak{D}_0 containing Ω , since a does not stabilize a lattice of the form $\mathcal{O}_K^2 + \pi^{-1} u \mathcal{O}_K$ with $u \in \mathcal{O}_K^2$ unless $u \in \mathcal{O}_K e_2 + \pi \mathcal{O}_K^2$. Then \mathfrak{D}_0 has valency 1 in $S_0(\Omega)$, and therefore, the

vertices $\mathfrak{D} \in \mathfrak{T}$ satisfying $\rho(\mathfrak{D}_0, \mathfrak{D}) \leq \rho(\mathfrak{D}_1, \mathfrak{D})$ are in $S_0(\Omega^{[t]})$ if and only if $\rho(\mathfrak{D}_0, \mathfrak{D}) \leq t$. We conclude that the vertices $\mathfrak{D} \in \mathfrak{T}$ satisfying $\rho(\mathfrak{D}_i, \mathfrak{D}) \leq \rho(\mathfrak{D}_{i+1}, \mathfrak{D})$ are in $S_0(\Omega)$ if and only if $\rho(\mathfrak{D}_i, \mathfrak{D}) \leq i$, and the result follows. \square

Corollary 4.2. *Let $\Omega = \mathcal{O}_K[a]$ where $a^2 = 0$. Then there is a conjugation taking $S_r(\Omega)$ to $S_0(\Omega)$ for every non-negative integer r .*

Example. Let $\mathfrak{H} \subseteq \mathfrak{D}_0$ be the order generated by $a_1 = \begin{pmatrix} 0 & 0 \\ \pi^2 & 0 \end{pmatrix}$ and $a_2 = \begin{pmatrix} 0 & \pi^3 \\ 0 & 0 \end{pmatrix}$. Then $S_0(\mathfrak{H})$ is the intersection of the branches

$$S_0(\mathcal{O}_K[a_1]) = \{\mathfrak{D} | \rho(\mathfrak{D}, \mathfrak{D}_i) \leq i + 2 \text{ for some } i \geq -2\},$$

$$S_0(\mathcal{O}_K[a_2]) = \{\mathfrak{D} | \rho(\mathfrak{D}, \mathfrak{D}_i) \leq 3 - i \text{ for some } i \leq 3\}.$$

It follows that $S_0(\mathfrak{H})$ is the set of orders at a distance not bigger than 2 from either \mathfrak{D}_0 or \mathfrak{D}_1 , and therefore $S_0(\mathfrak{H}) = S_0(\mathfrak{D}_0^{[2]} \cap \mathfrak{D}_1^{[2]})$.

Example. Assume K is a non-dyadic local field, and let \mathfrak{H} be an order generated by elements i, j satisfying $i^2 = j^2 = 1$ and $ij = -ji$. It follows from the results in this section that $\mathcal{O}_K[i]$ and $\mathcal{O}_K[j]$ are maximal paths. Note that $\mathbb{H} = \mathfrak{H}/\pi_K \mathfrak{H}$ is a matrix algebra, whence by Corollary 3.4 \mathfrak{H} is contained in a unique maximal order. In fact, it is not hard to prove that \mathfrak{H} is a maximal order. We conclude that the paths $S_0(\mathcal{O}_K[i])$ and $S_0(\mathcal{O}_K[j])$ intersect in a unique point.

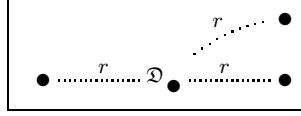
5 Admissible shapes for branches

Although the results in §3 can be obtained with no mention to the specific shape of the branch $S_r(\mathfrak{H})$, it turns out that there is only a very restricted set of possible branches. In order to prove this we need some preparation.

Lemma 5.1. *For any order \mathfrak{H} , the branches $S_r(\mathfrak{H})$ satisfies the following properties:*

1. *If $\mathfrak{D} \in S_r(\mathfrak{H})$ has two neighbors in $S_r(\mathfrak{H})$, one of which is in $S_{r+1}(\mathfrak{H})$, then $\mathfrak{D} \in S_{r+1}(\mathfrak{H})$.*

2. If $\mathfrak{D} \in S_0(\mathfrak{H})$, and there are three paths of length r in $S_0(\mathfrak{H})$ starting from \mathfrak{D} with no common edge, as in the following picture, then $\mathfrak{D} \in S_r(\mathfrak{H})$.



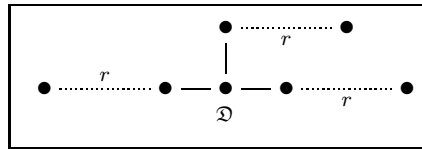
Proof. Note that any property of the form $\Phi(X) = [(Y \subseteq X) \implies (Z \subseteq X)]$ is stable under intersections. Recall now that $\mathfrak{D} \in S_r(\mathfrak{H})$ if and only if the closed r -ball $B[\mathfrak{D}; r] \subseteq \mathfrak{T}$ is contained in $S_0(\mathfrak{H})$, so both properties in the lemma are of the given type. Now the result follows by a straightforward case-by-case checking on the branches described in §4. \square

Lemma 5.2. *For any order \mathfrak{H} , the following properties hold:*

1. *If $S_{r+1}(\mathfrak{H})$ is not empty, then any maximal order in $S_r(\mathfrak{H})$ has a neighbor in $S_{r+1}(\mathfrak{H})$.*
2. *If $S_{r+1}(\mathfrak{H})$ is empty, then $S_r(\mathfrak{H})$ is a (possibly infinite) path.*

Proof. Assume $S_{r+1}(\mathfrak{H}) \neq \emptyset$. Let $\mathfrak{D} \in S_r(\mathfrak{H})$ and let $\mathfrak{D}' \in S_{r+1}(\mathfrak{H}) \subseteq S_r(\mathfrak{H})$. Since $S_r(\mathfrak{H})$ is connected, there is a path $\mathfrak{D}' = \mathfrak{E}_0 - \dots - \mathfrak{E}_n = \mathfrak{D}$ in $S_r(\mathfrak{H})$. Successive applications of the first part of the previous lemma prove that $\mathfrak{E}_0, \dots, \mathfrak{E}_{n-1}$ are all in $S_{r+1}(\mathfrak{H})$. First statement follows.

Assume now that $S_{r+1}(\mathfrak{H})$ is not a path. Then there exists a vertex $\mathfrak{D} \in S_r(\mathfrak{H})$ with three neighbors also in $S_r(\mathfrak{H})$. extending all of these lines by a path of length r heading in the opposite direction, as the picture shows, we obtain three segments of length $r + 1$ starting from \mathfrak{D} .



It follows from the second part of the previous lemma that $\mathfrak{D} \in S_{r+1}(\mathfrak{H})$ and the result follows. \square

We call a subset S of \mathfrak{T} an r -thick path if it consist of all vertices at a distance of at most r from some, finite or infinite, path. Next result is immediate from the preceding corollary by an obvious induction.

Proposition 5.1. *If an order \mathfrak{H} satisfies $S_r(\mathfrak{H}) \neq \emptyset$ and $S_{r+1}(\mathfrak{H}) = \emptyset$, then $S_k(\mathfrak{H})$ is an $(r - k)$ -thick path for every $0 \leq k \leq r$.*

Corollary 5.1. *If an order \mathfrak{H} is contained in finitely many maximal orders, then there exists an Eichler order \mathfrak{D} and a non-negative integer t such that $S_r(\mathfrak{H}) = S_r(\mathfrak{D}^{[t]})$ for every non-negative integer r . In particular, this is the case whenever \mathfrak{H} has maximal rank.*

Proposition 5.2. *If an order \mathfrak{H} satisfies $S_r(\mathfrak{H}) \neq \emptyset$ for every r , then $\mathfrak{H} = \mathcal{O}_K[a]$ for some nilpotent element a .*

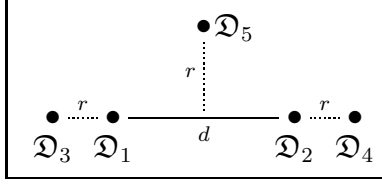
Proof. For every element $b \in \mathfrak{H}$, we have $\mathcal{O}_K[b] = \mathcal{O}_K[a]$ for some nilpotent element a by the computations in §4. Now, if \mathfrak{H} contain two linearly independent nilpotent elements, the explicit description of the branch $S_0(\mathcal{O}_K[a])$ plus the fact that the stabilizers of ends in the Bruhat-Tits tree are the borel subgroups of $\mathrm{GL}_2(K)$ (see [11], §II.1.3) show that the half infinite path used to construct $S_0(\mathcal{O}_K[a])$ and $S_0(\mathcal{O}_K[a'])$ cannot have the same end, and therefore both branches intersect in a finite set. The result follows. \square

Lemma 5.3. *An order \mathfrak{H} is the intersection of a finite family of maximal orders if and only if $\mathfrak{H} = \mathfrak{D}_0^{[r]}$ for some Eichler order \mathfrak{D}_0 and some non-negative integer $r \geq 0$.*

Proof. First we prove that if \mathfrak{H} is the intersection of a finite family of maximal orders, so is $\mathfrak{H}^{[t]}$ for every t . In fact, if $\mathfrak{H} = \bigcap_{\mathfrak{D} \in \Phi} \mathfrak{D}$, then $\mathfrak{H}^{[t]} = \bigcap_{\mathfrak{D} \in \Phi} \mathfrak{D}^{[t]}$ by Proposition 2.1. By Lemma 2.1, every order on the right of this identity is the intersection of a family of maximal orders. It follows that every order of the form $\mathfrak{H} = \mathfrak{D}_0^{[r]}$, where \mathfrak{D}_0 is an Eichler order, is the intersection of a family of maximal orders. On the other hand, if \mathfrak{H} is the intersection of a finite family of maximal orders, then $\mathfrak{H} = \mathfrak{D}_0^{[r]}$ by Corollary 5.1 and property (4) in Proposition 2.2. \square

Proof of Theorem 1.2 Note that all properties are local, and trivial at places ramifying \mathfrak{A} . We proceed locally. It is clear that (3) implies (2), and (2) implies (1) from the preceding lemma. Let $\mathfrak{H} = \mathfrak{D}^{[r]}$ for an Eichler order \mathfrak{D} of level d , say $\mathfrak{D} = \mathfrak{D}_1 \cap \mathfrak{D}_2$, where \mathfrak{D}_1 and \mathfrak{D}_2 are two maximal orders at distance d . Let \mathfrak{D}_3 , \mathfrak{D}_4 , and \mathfrak{D}_5 be maximal orders located as the picture

shows:



In other words, for instance, the path from \mathfrak{D}_5 to \mathfrak{D}_3 passes through \mathfrak{D}_1 , and $\rho(\mathfrak{D}_3, \mathfrak{D}_1) = r$. Let $\mathfrak{H}' = \mathfrak{D}_3 \cap \mathfrak{D}_4 \cap \mathfrak{D}_5$ and let $S = S_0(\mathfrak{H}')$. In particular S is a branch containing \mathfrak{D}_3 , \mathfrak{D}_4 , and \mathfrak{D}_5 . It follows from (1) in Lemma 5.1 that there is an r -deep vertex in the path joining \mathfrak{D}_1 and \mathfrak{D}_2 . Now reasoning as in the proof of Lemma 5.2, we see that both \mathfrak{D}_1 and \mathfrak{D}_2 , as much as every vertex in between, are r -deep in S . We conclude that $S \supseteq S_0(\mathfrak{H})$, whence $S = S_0(\mathfrak{H})$. Now $\mathfrak{H} = \mathfrak{H}'$ by (4) of Proposition 2.2, and the result follows. \square

Example. Let $\mathfrak{H} = \mathcal{O}_K[\pi^r i, \pi^s j]$ where i and j are as in the example at the end of §4. Then $S_0(\mathfrak{H})$ contains every vertex whose distance to the line $S_0(\mathcal{O}_K[i])$ is no larger than r and whose distance to the line $S_0(\mathcal{O}_K[j])$ is no larger than s . Let \mathfrak{D} be the unique maximal order containing i and j . Then $S_0(\mathfrak{H})$ contains the two vertices in $S_0(\mathcal{O}_K[i])$ at a distance s from \mathfrak{D} and the two vertices in $S_0(\mathcal{O}_K[j])$ at a distance r from \mathfrak{D} . In particular, if $r < s$, it follows as in the preceding proof that $S_0(\mathfrak{H}) = S_0(\mathfrak{E}^{[r]})$ for some Eichler order \mathfrak{E} of level $2(s - r)$.

6 Examples

Let \mathfrak{H} be a suborder of a global order $\mathfrak{D} = \mathcal{O}_X + I\mathfrak{D}'$, where \mathfrak{D}' is an Eichler order. Assume first $K\mathfrak{H} = L$ is a field. Note that the global spinor image $H(\mathfrak{D}|\mathfrak{H}) = \prod_{\varphi} H_{\varphi}(\mathfrak{D}|\mathfrak{H})$ contains the group $N_{L/K}(J_L)K^*$ corresponding to L , whence the representation field $F(\mathfrak{D}|\mathfrak{H})$ is a subfield of L . It follows that $F(\mathfrak{D}|\mathfrak{H})$ equals K^* unless the local spinor image $H_{\varphi}(\mathfrak{D}|\mathfrak{H})$ equals $N_{L_{\varphi}/K_{\varphi}}(L_{\varphi}^*)$ at every place φ . Since the spinor class field $\Sigma = \Sigma_{\mathfrak{D}} = \Sigma_{\mathfrak{D}'}$ is unramified, split completely at every place where \mathfrak{A} ramifies, and at every place where the level of \mathfrak{D}' is odd, we conclude that $F(\mathfrak{D}|\mathfrak{H}) = K^*$ unless $H_{\varphi}(\mathfrak{D}|\mathfrak{H}) = \mathcal{O}_{\varphi}^* K_{\varphi}^{*2}$ at every local place where L/K is inert. Note that for local unramified field extension L_{φ}/K_{φ} , the branch $S_0(\mathfrak{H}_{\varphi})$ of the order $\mathfrak{H}_{\varphi} = \mathcal{O}_{L_{\varphi}}^{[t]}$ is the set of vertices at a distance not exceeding t from the unique maximal order containing $\mathcal{O}_{L_{\varphi}}$. It follows that the diameter of $S_r(\mathfrak{H}_{\varphi})$ is $2(t - r)$. On the

other hand $S_r(\mathfrak{H}_\varphi)$ is infinite when $L = K\mathfrak{H}$ is not a field at φ , so that $H_\varphi(\mathfrak{D}|\mathfrak{H}) = K_\varphi^*$ in this case. Next result follows, generalizing the results in [5] or [8]:

Proposition 6.1. *In the preceding notations, When $L = K\mathfrak{H}$ is a quadratic extension of K , we have $F(\mathfrak{D}|\mathfrak{H}) = K$ unless the following conditions are satisfied:*

1. L is contained in the spinor class field Σ .
2. For every place φ inert for L/K , we have $\mathfrak{H}_\varphi = \mathcal{O}_{L_\varphi}^{[v_\varphi(I) + \frac{d_\varphi}{2}]}$ where v_φ denotes valuation at φ , while d_φ is the level at φ of the Eichler order \mathfrak{D}' .

In the latter case $F(\mathfrak{D}|\mathfrak{H}) = L$. On the other hand, if $L = K\mathfrak{H}$ is not a field, then $F(\mathfrak{D}|\mathfrak{H}) = K$.

Assume now that $K\mathfrak{H}$ is three-dimensional. This implies that $\mathfrak{A} = \mathbb{M}_2(K)$, and $K\mathfrak{H}$ is conjugated to the ring of matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. In particular, \mathfrak{H} is contained in a conjugate of the order $\mathfrak{H}_0 = \begin{pmatrix} \mathcal{O}_X & \mathcal{O}_X \\ 0 & \mathcal{O}_X \end{pmatrix}$. Since the branch $S_0(\mathfrak{H}_{0\varphi})$ is infinite at every place φ , next result follows:

Proposition 6.2. *If \mathfrak{H} is an order of rank 3, then $F(\mathfrak{D}|\mathfrak{H}) = K$.*

The situation is different for orders of rank 4. In fact, by choosing suborders of the type $\mathfrak{H} = \mathcal{O}_X + J\mathfrak{D}'$ where J is an ideal contained in I , we obtain that $F(\mathfrak{D}|\mathfrak{H})$ splits at every place dividing J/I . Next result follows:

Proposition 6.3. *For every field F with $K \subseteq F \subseteq \Sigma$ there exists an order \mathfrak{H} of rank 4, such that $F(\mathfrak{D}|\mathfrak{H}) = F$.*

Note that the diameter of $S_0(\mathfrak{D}_0^{[t]})$, where \mathfrak{D}_0 is an Eichler order of level d is $\rho = d + 2t$, and therefore the diameter of $S_r(\mathfrak{D}_0^{[t]})$ is $d + 2(t - r)$. Next result follows.

Proposition 6.4. *Let $\mathfrak{D} = \mathcal{O}_X + I\mathfrak{D}_1$ and $\mathfrak{H} = \mathcal{O}_X + J\mathfrak{D}_2$, where \mathfrak{D}_1 and \mathfrak{D}_2 are Eichler orders. Let $l_\varphi(\mathfrak{D}_i)$ denote the level at φ of the Eichler order \mathfrak{D}_i , then \mathfrak{H} embeds into an order in $\text{gen}(\mathfrak{D})$ if and only if at every place φ the following inequalities hold:*

$$v_\varphi(J) \geq v_\varphi(I), \quad l_\varphi(\mathfrak{D}_2) + 2v_\varphi(J) \geq l_\varphi(\mathfrak{D}_1) + 2v_\varphi(I).$$

Furthermore, if $\mathfrak{H} \subseteq \mathfrak{D}$, then $F(\mathfrak{D}|\mathfrak{H})$ is the largest subfield of $\Sigma(\mathfrak{D})$ splitting at every place where the second inequality is strict.

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